Generalized bit-moments and cumulants based on discrete derivative

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Abstract

We give a simple recipe based on the use of discrete derivative, to obtain generalized bit-moments obeying nonadditive statistics of Tsallis. The generalized bit-cumulants may be of two kinds, first which preserve the standard relations between moments and cumulants of a distribution, and are nonadditive with respect to independent subsystems. The second kind do not preserve usual moment-cumulant relations. These are additive in nature and Renyi entropy is naturally incorporated as the cumulant of order one.

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1. Introduction

The last decade saw an increasing interest in a nonextensive generalization of Boltzmann-Gibbs statistical mechanics, popularly called as Tsallis statistics. It is based on a nonadditive entropy, dependent on a real positive parameter q. Maximising Tsallis entropy under suitable constraints, leads to power-law type distributions which have been applied to various classes of physical systems and models, for example, fully developed turbulence, low-dimensional maps at onset of chaos, Hamiltonian systems with long-range interactions etc. (see Ref. [1] for a complete list of references.) Apart from these, it has motivated extension of the axiomatic foundations of information theory [2] as well as development of new mathematical tools and quantifiers [3].

In this spirit, the classical bit cumulants were generalized [4] within nonextensive thermodynamic approach. Bit statistics is a tool to describe the complicated probability distributions, such as generated by chaotic systems. Particularly, the second bit cumulant which is a generalization of specific heat, can be applied in the context of equilibrium and non-equilibrium phase transitions [5]. The q-generalized second bit cumulant has been applied [6] to study the issue of sensitivity to initial conditions in asymmetric unimodal maps at the threshold of chaos.

In this letter, we show that these generalized bit- moments and cumulants can be generated simply by introducing a discrete derivative in place of usual derivative into their definitions. The generalization we aim at can be achieved in two ways: (i) only generalize the bit-moments and then use the standard relations between bit-moments and bit-cumulants to define generalized cumulants, which yields q-cumulants of Ref. [4]; (ii) generalize the moments and cumulants independently from their basic definition (see below), using the discrete deriva-

tive. This procedure does not preserve standard relations between moments and cumulants.

First we briefly review the moments and cumulants in the Shannon-Gibbs bit statistics. Given a probability distribution $\{p_i|x = x_i\}_{i=1,\ldots,W}$ for a discrete random variable x, kth order moments of the distribution are defined as $M_k = \sum_{i=1}^{W} {x_i}^k p_i$. It is useful to define a generalized partition function, $Z(\sigma) = \sum_i p_i e^{\sigma x_i}$. Then moments of the distribution $\{p_i\}$ are given as

$$M_k = \frac{\partial^k}{\partial \sigma^k} Z(\sigma)\Big|_{\sigma=0}. \tag{1}$$

Similarly, the cumulants for the distribution may be obtained from a generating function $G(\sigma) = \ln Z$, as follows

$$C_k = \frac{\partial^k}{\partial \sigma^k} G(\sigma)\Big|_{\sigma=0}. \tag{2}$$

Let us take x to be the fluctuating bit number, $-\ln\, p_i \equiv -a_i$. Thus the relevant partition function is

$$Z(\sigma) = \sum_i p_i e^{-\sigma a_i}. \tag{3}$$

Now within the above statistical analysis, the first cumulant C_1 (which is also equal to first moment M_1) is the average value of the bit number. In the present context, it is equal to Shannon entropy. The second cumulant is given by $C_2 = \sum_i (\ln\, p_i)^2 p_i - (\sum_i p_i \ln\, p_i)^2$, and is called the bit variance. Realizing that within Tsallis' framework, the generalized bit number can be given by

$$-[a_i] = \frac{(p_i)^{-\Delta} - 1}{\Delta}, \tag{4}$$

where from now on we imply $\Delta = (1-q)$, a generalization [4] of standard cumulants was considered by modifying the partition function in Eq. (3) based on $[a_i]$ instead of a_i. Such a generalization of second bit cumulant has found an interesting application in low-dimensional maps at chaotic threshold. Note that in

applications of Tsallis formalism to such systems a critical value of $q = q^*$ exists, which may be inferred from the study of sensitivity to initial condition [7] or from the multifractal spectrum of the attractor at chaotic threshold [8]. Although, a priori, q introduced above is a free parameter, it is found to behave like the critical index q^* in quantifiers such as the generalized second bit cumulant. This implies that [6] information about relation of Tsallis index q^* and inflexion parameter z of the nonlinear mapping can be reliably obtained, where it is difficult to do so by other techniques (see also [9]).

2. Method (i)

In the following, we show that moments of the above bit number in Eq. (4) can be obtained by modifying the derivative rule in the Eq. (1), but keeping the partition function of Eq. (3) unchanged. We simply replace the ordinary derivative with respect to σ by a discrete derivative operator defined as follows

$$\partial_{\Delta,x} f(x) = \frac{f(x+\Delta) - f(x)}{\Delta}. \tag{5}$$

Naturally, as $\Delta = (1-q) \to 0^+$, we get back the ordinary derivative. Now we make the substitution

$$\frac{\partial^k}{\partial \sigma^k} Z(\sigma) \to \partial^k_{\Delta,\sigma} Z(\sigma). \tag{6}$$

In other words, the q-moments of order k, are defined as

$$M_k^{(q)} = \partial^k_{\Delta,\sigma} Z(\sigma)|_{\sigma=0}. \tag{7}$$

It can be easily verified that the q-moment of order 1 is

$$M_1^{(q)} = \frac{1 - \sum_i (p_i)^q}{q-1}, \tag{8}$$

which is Tsallis entropy. Pseudoadditivity of Tsallis entropy can be understood as originating from the modified Leibnitz rule

$$\partial_{\Delta,x} f g(x) \quad = \quad \partial_{\Delta,x} f(x) g(x) + f(x+\Delta) \partial_{\Delta,x} g(x) \tag{9}$$

$$= \quad \partial_{\Delta,x} f(x) g(x+\Delta) + f(x) \partial_{\Delta,x} g(x). \tag{10}$$

In general, we have

$$M_k^{(q)} = \sum_i (-[a_i])^k p_i. \tag{11}$$

Note that generalized bit number $-[a_i]$ of Eq. (4) appears automatically by operating discrete derivative on the "classical" partition function, Eq. (3). To obtain q-cumulants, we have to consider the rule in Eq. (6) again. For that purpose we note that

$$C_1 = \frac{1}{Z} \frac{\partial}{\partial \sigma} Z(\sigma) \Big|_{\sigma=0}. \tag{12}$$

Thus q-cumulant of order 1 can be written as

$$C_1^{(q)} = \frac{1}{Z} \partial_{\Delta,\sigma} Z(\sigma) \Big|_{\sigma=0}. \tag{13}$$

Due to the normalization $Z(\sigma = 0) = 1$, $C_1^{(q)} = M_1^{(q)}$. Similarly, we have

$$C_2 = \frac{1}{Z} \frac{\partial^2}{\partial \sigma^2} Z(\sigma) \Big|_{\sigma=0} - \frac{1}{Z^2} \left(\frac{\partial Z(\sigma)}{\partial \sigma} \right)^2 \Big|_{\sigma=0}. \tag{14}$$

Applying rule of Eq. (6), we have

$$C_2^{(q)} = \frac{1}{Z} \partial^2_{\Delta,\sigma} Z(\sigma) \Big|_{\sigma=0} - \frac{1}{Z^2} (\partial_{\Delta,\sigma} Z(\sigma))^2 \Big|_{\sigma=0}, \tag{15}$$

and so on for higher order q-cumulants. Note that $C_2^{(q)}$ is the variance of the generalized bit number $-[a_i]$ in the same way as C_2 is the variance of the classical bit number $-\ln\, p_i$. In the above scheme, we have effectively generalized the standard bit moments to q-moments and then q-cumulants follow as a result of standard relations between moments and cumulants of a distribution. In this way, we maintain the usual meaning of moments and cumulants of a distribution.

3. Method (ii)

Naturally, one cannot claim a unique way to define the generalized bit moments as well as the cumulants. This problem is similar to the one of defining

a (multi)parameter dependent generalization of Shannon entropy, where some physical/mathematical principle is required to motivate the generalization. For example, seeking the form invariance of certain mathematical relations can require the valid generalized entropy to be an appropriate renormalisation of Tsallis entropy [10]. In this sense, our method (i) is motivated by the requirement of keeping the standard meaning of bit moments and cumulants, and also the relations between them. Otherwise, the use of a modified derivative operator offers at least one more possibility to define a generalization of the standard bit cumulants. The new rule may be given as

$$\frac{\partial^k}{\partial \sigma^k} \to \partial^k_{\Delta,\sigma}. \tag{16}$$

Compare this with Eq. (6). Then applying the above rule to Eqs. (1) and (2), we note that q-moments remain the same as given by Eq. (7), but q-cumulants are a new set of quantities given by

$$\mathcal{C}_k^{(q)} = \partial^k_{\Delta,\sigma} G(\sigma)|_{\sigma=0}. \tag{17}$$

This generalization seems interesting due to the fact that the first q-cumulant is

$$\mathcal{C}_1^{(q)} = \frac{\ln\, \sum_i (p_i)^q}{1-q}, \tag{18}$$

which is Renyi entropy with index q [11]. *This distinguishes Tsallis entropy and Renyi entropy, based on the use of discrete derivative: Tsallis entropy is obtained from Z, while Renyi entropy follows from the generating function $G = \ln Z$.* Also this definition of Renyi entropy potentially connects it to non-commutative differential calculus [12] which is based on discrete derivative.

However, it should be pointed out that in method (ii), the standard relations between moments and cumulants cannot be preserved. Thus as is already evident, the first q-cumulant is not equal to first q-moment. This implies that Renyi

entropy cannot be inferred here as a simple expectation value of some generalized bit number. In fact, it can be written only as a kind of nonlinear average [13].

Similarly, the q-cumulant of order 2 is given by

$$\mathcal{C}_2^{(q)} = \frac{1}{(1-q)^2}\left[\ln \sum_i (p_i)^{2q-1} - 2\ln \sum_i (p_i)^q\right], \tag{19}$$

which is not equal to the bit variance of generalized bit number $-[a_i]$, given from Eq. (15) as

$$C_2^{(q)} = \frac{1}{(1-q)^2}\left[\sum_i (p_i)^{2q-1} - (\sum_i (p_i)^q)^2\right]. \tag{20}$$

Thus *there does not appear to be a generalized bit number connected with Renyi entropy, unlike $-[a_i]$ for Tsallis entropy.* However, these quantities do go back to the standard bit variance in the limit $q \to 1$. The positivity of $\mathcal{C}_2^{(q)}$ follows from the similar property of $C_2^{(q)}$. Thus if $\mathcal{C}_2^{(q)}$ is considered as generalization of $C_2^{(1)}$, then it is interesting to see the implications of using $\mathcal{C}_2^{(q)}$ instead of $C_2^{(q)}$ in such systems as, for example, low dimensional maps along the lines of Ref. [6]. Such a comparative study can help to elucidate the separate roles of Renyi and Tsallis entropies in these systems when they exhibit power-law behaviour [14].

After observing that discrete derivative may be relevant for Renyi entropy, we consider the special case of multifractals [15]. For such measures also, we can define a generalized partition function $Z(\beta) = \sum_i (p_i)^\beta$, where the sum ranges over boxes of size l. We note that $Z(\beta)$ and $Z(\sigma)$ in Eq. (3) are equivalent and $\beta \equiv 1-\sigma$. Thus discrete derivative can be expected to be significant for multifractal measures also. For length scale $l \to 0$, the partition function scales as

$$Z(\beta) = \sum_i (p_i)^\beta \sim l^{-\tau(\beta)}. \tag{21}$$

The measure is characterized by a whole sequence of exponents $\tau(\beta)$, given by

$$\tau(\beta) = -\lim_{l\to 0} \frac{\ln \sum_i (p_i)^\beta}{\ln l}. \tag{22}$$

Choosing different values of β helps to scan subsets with different probability strengths scaling as $p_i \sim l^{\alpha_i}$. These properties can be expressed by taking the derivative of $\tau(\beta)$ as

$$\frac{d}{d\beta}\tau(\beta) = -\lim_{l\to 0}\frac{\sum_i (p_i)^\beta \ln p_i}{\sum_i (p_i)^\beta \ln l}. \tag{23}$$

Let us replace the ordinary derivative by the discrete derivative and taking into the fact that $\beta = 1-\sigma$, we find that

$$\partial_{\Delta,\beta}\tau(\beta) = -\lim_{l\to 0}\ \ln\left(\frac{\sum_i (p_i)^{\beta-\Delta}}{\sum_i (p_i)^\beta}\right)\Big/\Delta \ln l. \tag{24}$$

We find that

$$\partial_{\Delta,\beta}\tau(\beta)|_{\beta\to\infty} = -\alpha_{\max}, \tag{25}$$

and

$$\partial_{\Delta,\beta}\tau(\beta)|_{\beta\to-\infty} = -\alpha_{\min}, \tag{26}$$

where $\alpha_{\max}$ and $\alpha_{\min}$ are the end points of the multifractal spectrum. These results are just the same as will be obtained by working with Eq. (23). This is reasonable, as discrete derivative in β measures the change in function with respect to a finite increment Δ. As $|\beta| \to \infty$, any finite increment Δ is infinitismally small relative to the magnitude of $|\beta|$ and so we expect discrete derivative to yield same result as the usual derivative in this limit. However, for finite β such as equal to unity, we have

$$\partial_{\Delta,\beta}\tau(\beta)|_{\beta=1} = \frac{1}{(q-1)}\lim_{l\to 0}\frac{\ln\ \sum_i (p_i)^q}{\ln l} = D_q, \tag{27}$$

which is the definition of generalized dimension. This again illustrates that discrete derivative is the generator of Renyi information, where using the ordinary derivative would give Shannon information

$$\left.\frac{d}{d\beta}\tau(\beta)\right|_{\beta=1} = \lim_{l\to 0}\sum_i \frac{p_i \ln p_i}{\ln l}. \tag{28}$$

4. Concluding remarks

We have used discrete derivative to obtain generalized bit moments or cumulants and within that, we have noticed the basis to distinguish Tsallis and Renyi entropy. Tsallis entropy has been previously related to Jackson's q-derivative and this observation motivated the search for connections between quantum group theory and nonextensive statistics [16]. In fact, it is well known that discrete derivative and Jackson derivative can be mapped onto each other [12]. Briefly, the discrete translation produced by the former operator correspond to dilatation by the latter. Thus the modification of the derivative in Eq. (6) can be written equivalently as

$$\frac{\partial^k}{\partial \sigma^k} Z(\sigma) \to (-y D_{y,q})^k Z(y), \tag{29}$$

where $D_{y,q}$ is the Jackson derivative operator

$$D_{y,q} f(y) = \frac{f(qy) - f(y)}{(q-1)y}. \tag{30}$$

The variable y is defined as $y = e^{-\sigma}$. Note that the operator $(-y D_{y,q})^k$ goes to $\frac{\partial^k}{\partial \sigma^k}$ as $q \to 1$.

Secondly, as both Shannon and Renyi entropies are additive with respect to independent subsystems, and nonadditivity has been thought to be related to Jackson derivative, it seems Renyi entropy may not have any connection with discrete derivatives. This is somewhat surprising as Jackson derivative seems relevant for multifractal sets [17] and Renyi entropy has also proved useful for such systems [18]. After showing above that discrete derivative is the generator of Renyi entropy, we point out that it is also consistent with the additive property of Renyi entropy which has been derived here from the generating function $G = \ln Z$. Thus for independent subsystems A and B, $Z(A+B) = Z(A).Z(B)$ and so the generating function is additive $G(A+B) = G(A) + G(B)$. Now the discrete

derivative is distributive:

$$\partial_{\Delta,x}(f+g)(x) = \partial_{\Delta,x}f(x) + \partial_{\Delta,x}g(x). \tag{31}$$

and additivity of Renyi entropy follows directly from this.

In general, we observe that cumulants of method (i) are nonadditive with respect to independent subsystems, whereas those of method (ii) are additive in nature. Thus based on discrete derivative, for one class of (nonextensive) q-bit moments in Eq. (7), we have two classes of q-bit cumulants distinguished by their additive properties. Tsallis and Renyi entropies have been cast here in the more general framework of bit-moments and cumulants. Particularly, the concept of biased averaging $\sum_i (p_i)^q$ found in both the entropies is shown to have its origin in the finite increment induced by discrete derivative or equivalently, dilatation induced by Jackson derivative. It is hoped that the present approach will help in better understanding the origin and connection between these entropies.

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